Background

- Towers of Hanoi: a classic problem
  - Three poles to move disks on
  - Goal is to move all disks to third pole
  - Certain rules apply in movement of disks
- Can be represented as a direct graph
  - $S =$ start node, $A =$ auxiliary node (between $S$ and $D$ nodes), $D =$ destination node
  - Two edges between every two nodes
Solvable and Finite Graphs

- A graph is *solvable* when:
  - There exist vertices S, D, and A
  - There exist paths from S to A, from A to D, and from D to S

- *Solvable* means arbitrary number of disks can be moved from S to D

- Any graph that is not solvable is *finite*
  - A finite number of disks can be moved
Hanoi Graphs

- Original Hanoi graph requires $2^d - 1$ moves ($d =$ number of disks)

- Modified Hanoi graph (no edges between S and D) requires $3^d - 1$ moves
The K Graph

- Has $k+3$ nodes, $S_0$ through $S_k$, A, D
- Established formula: $k \times d + 3^d - 1$
- Algorithm: moving from to $S_k$ takes $k \times d$ moves, then a modified Hanoi graph
Conjecture

- The K graph is the worst graph\textsuperscript{[2]}
- Possibly false
- Looking for a counterexample
- The Cycle graph a possibility
The Cycle Graph

- 1 edge between each 2 nodes (a “cycle”)
- Two scenarios: 1. $d < n$; 2. $d \geq n$
Scenario 1 Formula

- Starting with first scenario, \( d < n \)
- Formula: \( 0.5n^3 - (n - d - 0.5)n^2 + [-1 + (n - d)\text{choose}(2)]n + n - d \)
- Simplifies to: \( n\left[(d+1)\text{choose}(2)\right] - d \)
- Further simplifies to: \( n^*\left[(d+1)\text{choose}(2)\right] - d \)
Algorithm: Step One

- Found algorithm
- Spread disks out on graph
- Move smallest disk to last or \( (n^{th}) \) node
- Move second smallest disk to second-to-last or \( (n-1)^{th} \) node
- Until last disk is on \( (n-d+1)^{th} \) node
Figure: Step One

For example: $n = 4, d = 3$
Algorithm: Step Two

- Move the disks forward one by one
- Move the smallest disk to first node
- Move the second smallest to last node
- Whenever largest disk in-game reaches last node, remove it from game
- Until all disks are removed
Figure: Step Two

- Same example
### Counterexample

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<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
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### Table 1. Partial table for number of moves for the K graph when d < n

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<th>Number of disks (d)</th>
<th>Number of nodes (n)</th>
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<th>5</th>
<th>6</th>
<th>7</th>
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### Table 2. Partial table for the cycle graph when d < n
Scenario 2

- When \( d \geq n \)
- Still writing an algorithm
- Basic idea:
  - Spread out first \( n-1 \) disks
  - Greedy algorithm: make least moves at each node to make empty node for previous disk
  - Becomes easy when \( d < n \)
Conclusions

- K graph requires less moves than cycle graph when $d < n$

- Conjecture
  - K graph = worst graph in terms of moves
  - Proven false
  - Do not need to look at case when $d \geq n$
Sources

1. Leiss, Ernst L. “Solving the ‘Towers of Hanoi’ on graphs.”
2. Leiss, Ernst L. “The Towers of Hanoi on Graphs: Upper and Lower Bounds on the Number of Moves.”
Questions

Thank you. Any questions?